Solution Sheet 1

Exercise 1. (for credit, due on 21 September) Prove that the Riemann surface $\mathbb{C}/\mathbb{Z}$ is biholomorphic to $\mathbb{C}\backslash\{0\}$.

Solution 1. Let $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ be the quotient map and define

$$f : \mathbb{C} \to \mathbb{C}\backslash\{0\}, \quad f(z) = e^{2\pi iz}.$$

Because $f(z+n) = f(z)$ for all $z \in \mathbb{C}$ and $n \in \mathbb{Z}$, f is constant on $\mathbb{Z}$-orbits. Thus f induces a continuous map

$$\overline{f} : \mathbb{C}/\mathbb{Z} \to \mathbb{C}\backslash\{0\}, \quad \overline{f}([z]) = e^{2\pi iz},$$

with $f = \overline{f} \circ \pi$. We explicitly check that the map $\overline{f}$ is holomorphic. Fix $[z_0] \in \mathbb{C}/\mathbb{Z}$ and choose a disk $U = D(z_0, r) \subset \mathbb{C}$ with $0 < r < \frac{1}{2}$. Then the translates $U + n$ are pairwise disjoint, so $V = \pi(U)$ is open and the restriction $\pi|_U : U \to V$ is a biholomorphism. Hence on V we can write $\bar{f} = f \circ (\pi|_U)^{-1}$ as a composition of holomorphic maps, thus $\bar{f}$ is holomorphic. As $[z_0]$ was arbitrary, $\bar{f}$ is holomorphic on all of $\mathbb{C}/\mathbb{Z}$. The map $\overline{f}$ is injective: if $\overline{f}([z]) = \overline{f}([w])$, then $e^{2\pi iz} = e^{2\pi iw}$ if and only if $z - w \in \mathbb{Z}$, i.e. $[z] = [w]$. To show surjectivity, pick for any $w \in \mathbb{C}\backslash\{0\}$ a local branch *log* of the logarithm near w. Then with $z = \frac{1}{2\pi i}\log w$ it holds that $e^{2\pi iz} = w$, hence $w = \overline{f}([z])$. Since different choices of the branch of log differ by $2\pi ik$ with $k \in \mathbb{Z}$, the class $[z]$ does not depend on the branch and is well defined. We conclude that $\overline{f}$ is bijective and holomorphic. We will later prove that a bijective holomorphic map between Riemann surfaces is automatically a biholomorphism. To make the inverse explicit, we give a local holomorphic inverse map to $\overline{f}$. For $w \in \mathbb{C}\backslash\{0\}$, pick a domain $U \subset \mathbb{C}\backslash\{0\}$ containing w on which a holomorphic branch $\log_U$ exists. We define

$$g : U \to \mathbb{C}/\mathbb{Z}, \quad g(w) = \left[\frac{\log_U w}{2\pi i}\right].$$

The map g is holomorphic on U and $\overline{f} \circ g = \mathrm{id}_U$. Thus, $\overline{f}$ locally admits a holomorphic inverse map, and these glue to a global holomorphic inverse.

Exercise 2. Prove that the unit disk $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$ and the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \ : \ \Im(z) > 0\}$ are biholomorphic.

Solution 2. Define $f(z) = i\frac{1+z}{1-z}$ and $g(w) = \frac{w-i}{w+i}$. Note that f is holomorphic on $\mathbb{D}$ and g is holomorphic on $\mathbb{H}$. For $w = x + iy$ with $y > 0$ we have

$$\left|\frac{w-i}{w+i}\right|^2 = \frac{x^2 + (y-1)^2}{x^2 + (y+1)^2} < 1,$$

because $(y+1)^2 > (y-1)^2$ when $y > 0$. Thus $g(\mathbb{H}) \subset \mathbb{D}$. If $|z| < 1$, then

$$\Im\left(i\frac{1+z}{1-z}\right) = \frac{1-|z|^2}{|1-z|^2} > 0.$$

Hence $f(\mathbb{D}) \subset \mathbb{H}$. One can check that f and g are mutual inverses by direct computation. Therefore $g : \mathbb{H} \to \mathbb{D}$ is a biholomorphism.

Exercise 3. (challenging)
Let $\mathcal{S} = \{f : \mathbb{D} \to \mathbb{C}$ holomorphic, injective, $f(0) = 0, f'(0) = 1\}$.

(a) Show that the map

$$k : \mathbb{D} \to \mathbb{C} \setminus (-\infty, -1/4], \quad z \mapsto \frac{z}{(1-z)^2}$$

is a biholomorphism and $k \in \mathcal{S}$.

(b) If $f(z) = z + \sum_{n\geq 2} a_n z^n \in \mathcal{S}$, prove that $|a_2| \leq 2$ and that equality holds for the function k.

(c) Prove that every $f \in \mathcal{S}$ satisfies $B_{1/4}(0) \subset f(\mathbb{D})$ and that the constant $1/4$ cannot be increased.

Remark. The Bieberbach conjecture, proved in 1985 by de Branges, states that if $f(z) = z + \sum_{n\geq 2} a_n z^n \in \mathcal{S}$, then $|a_n| \leq n$ for all $n \geq 2$.

Solution 3. Look up a proof on the internet.

Exercise 4. Let X and Y be Riemann surfaces. Assume that X is connected and compact.

(a) Let $f : X \to Y$ be a holomorphic and non-constant map. Show that f surjects onto a compact connected component of Y.

(b) Deduce from (a) that every holomorphic map $f : X \to \mathbb{C}$ is constant.

(c) Deduce from (a) the fundamental theorem of algebra: a non-constant complex polynomial has at least one root.

Solution 4.

(a) Since f is holomorphic and non-constant, the open mapping theorem implies that $f(X)$ is open. Because f is continuous and X compact, the image $f(X)$ is compact, hence also closed because Y is Hausdorff. As X is connected, $f(X)$ is connected. Let C be the connected component of Y containing $f(X)$. As $f(X)$ is nonempty and both open and closed, this forces $f(X) = C$.

(b) The existence of a non-constant holomorphic map $f : X \to \mathbb{C}$ contradicts part (a) since $\mathbb{C}$ is not compact.

(c) A polynomial p determines a holomorphic map $F : \mathbb{CP}^1 \to \mathbb{CP}^1$ by

$$F(z) = p(z) \text{ on } \mathbb{C}, \quad F(\infty) = \infty.$$

This map is non-constant and holomorphic. Part (a) implies that it is onto, hence $0 \in \operatorname{Im}(F)$, that is, there exists $z \in \mathbb{C}$ with $p(z) = 0$.

Exercise 5. Let $\Omega \subseteq \mathbb{C}$ be an open subset of the complex plane. A meromorphic function on Ω can be defined as a function f which is holomorphic outside a discrete subset $S = \{p_i,\ i \in I\} \subseteq \Omega$ and admits the following Laurent expansion in a neighborhood of p_i ($i \in I$):

$$f(z) = \frac{a_{-N}}{(z-p_i)^N} + \ldots + \frac{a_{-1}}{z-p_i} + a_0 + a_1(z-p_i) + \ldots \qquad (1)$$

(1) Show that a meromorphic function on Ω is the same as a holomorphic function $\Omega \to \mathbb{CP}^1$ (which is not identically ∞ on any connected component).

(2) A function of the form $R(z) = P(z)/Q(z)$, where P and Q are polynomials, is called a rational function. Show that a rational function defines a meromorphic function on $\mathbb{CP}^1$.

(3) Prove that every meromorphic function on $\mathbb{CP}^1$ is rational.

Solution 5. (1) Let f be a meromorphic function on $\Omega \subseteq \mathbb{C}$ with set of poles S. We extend $f : \Omega \backslash S \to \mathbb{C}$ to $\overline{f} : \Omega \to \mathbb{P}^1_{\mathbb{C}}$ by setting $\overline{f}(p) = \infty$ for $p \in S$. This defines a holomorphic map. Indeed, in a neighborhood of $p \in S$, by (1) we have

$$\frac{1}{f(z)} = \frac{(z-p)^N}{a_{-N} + a_{-N+1}(z-p) + \ldots}.$$

This function is holomorphic on a punctured neighborhood of p and has a zero of order N at p. Therefore it extends holomorphically across p.
Conversely, let $g : \Omega \to \mathbb{P}^1_{\mathbb{C}}$ be holomorphic and not constantly equal to ∞. Then $S := g^{-1}(\infty) \subseteq \Omega$ is discrete. The function $g : \Omega \backslash S \to \mathbb{C}$ is holomorphic. Near

any $p \in S$, the function $1/g$ is holomorphic. Hence for any $p \in S$, there exists a neighborhood U_p, an integer $N_p \geq 1$, and a holomorphic map h_p on U_p with $h_p(p) \neq 0$ such that

$$\frac{1}{g(z)} = (z-p)^{N_p} h_p(z) \quad \text{for } z \in U_p$$

It follows that $g(z)$ has a pole of order N_p at p, i.e. it admits a Laurent expansion of the prescribed form.

(2) As a quotient of two holomorphic functions, R defines a meromorphic function on $\mathbb{C}$. Hence R yields a holomorphic map $R : \mathbb{C} \to \mathbb{CP}^1$. We extend R to $\overline{R} : \mathbb{CP}^1 \to \mathbb{CP}^1$ by setting $\overline{R}(\infty) = \lim_{z\to\infty} P(z)/Q(z)$. If we denote by a (resp. b) the leading coefficients of P (resp. Q), then we can write

$$\overline{R}(\infty) = \begin{cases} 0 & \text{, if } \deg(R) < 0 \\ \frac{a}{b} & \text{, if } \deg(R) = 0 \\ \infty & \text{, if } \deg(R) > 0 \end{cases}$$

One can check that these values correspond to $P(1/z)/Q(1/z)$ evaluated at 0. Hence $\overline{R}$ is holomorphic and thus defines a meromorphic function on $\mathbb{CP}^1$.

(3) Let $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be a holomorphic map, which is not constantly equal to ∞. Consider the discrete set $S = f^{-1}(\infty) \subseteq \mathbb{P}^1_{\mathbb{C}}$. Since $\mathbb{CP}^1$ is compact, S is finite. Let the finite poles be $\{p_1, \cdots, p_m\} \in \mathbb{C}$ with orders N_j and write their principal part as

$$(P_{p_j} f)(z) = \sum_{k=1}^{N_j} \frac{a_{j,k}}{(z-p_j)^k}.$$

In the neighborhood of ∞ we use the local coordinate $u = 1/z$ and then we have the Laurent expansion

$$f(1/u) = \sum_{k=-N_\infty}^{\infty} a_{\infty,k}\, u^k$$

for some $N_\infty \geq 0$. The terms with negative powers of u correspond to the polynomial

$$P(z) = \sum_{k=1}^{N_\infty} a_{\infty,-k}\, z^k.$$

Now we set

$$R(z) = \sum_{j=1}^{m} (P_{p_j} f)(z) + P(z);$$

this is a rational function. The function $g(z) = f(z) - R(z)$ is by construction holomorphic on $\mathbb{CP}^1$, and its image lies in $\mathbb{C}$. By Exercise 3, a holomorphic function $\mathbb{CP}^1 \to \mathbb{C}$ is constant. Hence we conclude that f is given by a rational function.